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AN ALGORITHM FOR SINGULAR QUADRATIC PROGRAMMING

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ABSTRACT

The present paper deals with an algorithm for quadratic programming when the matrix of the quadratic part of the maximand is semi-definite. When m is the number of linear inequality constraints the algorithm leads to a simplex tableau of order $m \times 2m$.

In Section 1 an algorithm is given which is valid when the matrix of the quadratic part is strictly definite. It was originally proposed in [1], but is restated here in a self-contained way. Section 2 deals with linear programming, which is considered as a quadratic programming problem with a zero matrix for the quadratic part of the maximand. It is an introduction to Section 3, dealing with singular quadratic programming.

The algorithm solves explicitly for the dual problem as well.

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1. Quadratic Programming

The problem is to maximize the function

$$(1.1) \qquad Q^*(x) = a'x - \frac{1}{2}x'Bx \qquad \left[\sum_{i=1}^{n} a_i x_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_i x_j\right]$$

where B is a symmetric, positive definite matrix, subject to

$$(1.2) \qquad C'x \leqq d \qquad \left[\sum_{i=1}^{n} c_{hi}x_i \leqq d_h\ ; \quad h = 1, \dots, m\right]$$

or, equivalently,

$$(1.3) \qquad C'x + Iw = d, \quad w \geqq 0 .$$

The m-vector d will be called the source-vector; the m-vector w will be called the slack-vector. Either the slack is zero, and the corresponding source is fully used; or else the slack is positive, and

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the corresponding source is amply available. Non-negativity conditions may be included in (1.2) or (1.3). The first n inequalities of (1.2) are then $-Ix \leqq 0$; the first n elements of the slack-vector are then exactly the elements of x.

In Theorems 1 and 2 we will derive the Kuhn-Tucker conditions. Consider:

$$Q(x, u) = a'x - \frac{1}{2}x'Bx - u'[C'x + Iw - d] \,, \tag{1.4}$$

where the m-vector u will be called the Lagrangean vector. Then:

<u>Theorem 1</u>: A <u>sufficient</u> condition for x^* to solve the problem (1.1) - (1.2) is:

i) $C'x^* \leqq d$ [or, equivalently, $C'x^* + Iw^* = d$; $w^* \geqq 0$],

ii) $\frac{dQ}{dx}\Big|_{x=x^*} = 0$ [i.e. $a - Bx^* - Cu^* = 0$]

iii) $u^* \geqq 0$

iv) $u^{*\prime}w^* = 0$.

<u>Proof</u>: Suppose $\bar{x}$ is a vector such that $C'\bar{x} \leqq d$ (or $C'\bar{x} + I\bar{w} = d$, $\bar{w} \geqq 0$). Then:

$$(1.5)\qquad Q^*(x^*) - Q^*(\bar{x}) = a'(x^* - \bar{x}) - \frac{1}{2}x^{*\prime}Bx^* + \frac{1}{2}\bar{x}'B\bar{x}$$

$$= a'(x^* - \bar{x}) + \frac{1}{2}(\bar{x} - x^*)'B(\bar{x} - x^*) - x^{*\prime}Bx^* + x^{*\prime}B\bar{x}$$

$$= (a' - x^{*\prime}B)(x^* - \bar{x}) + \frac{1}{2}(\bar{x} - x^*)'B(\bar{x} - x^*)$$

$$= u^{*\prime}C'(x^* - \bar{x}) + \frac{1}{2}(\bar{x} - x^*)'B(\bar{x} - x^*)$$

$$= u^{*\prime}(d - w^* - C'\bar{x}) + \frac{1}{2}(\bar{x} - x^*)'B(\bar{x} - x^*)$$

$$= u^{*\prime}(d - C'\bar{x}) + \frac{1}{2}(\bar{x} - x^*)'B(\bar{x} - x^*)$$

$$= u^{*\prime}\bar{w} + \frac{1}{2}(\bar{x} - x^*)B(\bar{x} - x^*) \geq 0\,,$$

since $u^* \geqq 0$, $\bar{w} \geqq 0$, and B is positive definite. The equality only holds if $\bar{x} = x^*$.

<u>Theorem 2</u>: A <u>necessary</u> condition for x^* to solve (1.1) - (1.2) is

i) $C'x^* \leqq d$ [or, equivalently, $C'x^* + Iw^* = d$, $w^* \geq 0$]

ii) $\frac{dQ}{dx}\Big|_{x=x^*} = 0$ [i.e. $a - Bx^* - Cu^* = 0$]

iii) $u^* \geqq 0$

iv) $u^{*\prime}w^* = 0$.

Proof: i) is obvious. ii) is the well-known Lagrangean condition for maximizing under constraints. As for iii), consider

$$\frac{dQ}{dd} = u \,, \tag{1.6}$$

which implies that, if u has a negative element, a decrease in its associated source d (or, equivalently, an increase in the associated slack w) will increase the value of $Q = Q^*$. There is nothing in the nature of the relevant inequality of (1.2) preventing this decrease. Hence, no element of u can be negative[1]. To prove iv), suppose $u_h > 0$. Then, vide (1.6), an increase in d_h, which is feasible as long as $w_h > 0$, increases Q. Hence, if $u_h > 0$, $w_h > 0$, there is no maximum. Conversely, suppose w_k is positive. Then, if $u_k > 0$, since

$$\frac{dQ}{dw} = -u \,, \tag{1.7}$$

a decrease in w_k increases Q. Hence, no maximum.

The two theorems lead to the following approach. From

$$a - Bx - Cu = 0 \tag{1.8}$$

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[1] Actually, this argument glosses over a subtle point. For it requires the vector u to be defined! Using (1.10) below we see that

$$u = (C'_w B^{-1} C_w)^{-1}(d - C'_w B^{-1} a) \,,$$

where C'_w is the submatrix of C' consisting of all rows with slack zero. If the rows of C'_w are dependent, u cannot be uniquely solved. In some cases, this can indeed lead to difficulties.

and

$$C'x + Iw = d \tag{1.9}$$

we derive

$$-C'B^{-1}Cu + Iw = d - C'B^{-1}a \tag{1.10}$$

by solving (1.8) for x and substituting the result in (1.9). The problem is to find a non-negative solution (u^*, w^*) to this system of m equations in $2m$ unknowns such that $u^{*\prime}w^* = 0$. As initial solution we can clearly take[2] $w = d - C'B^{-1}a$. If $w \geq 0$, then we have the solution. If not, we replace a negative w_i by u_i (which u_i will then be positive, see below). Thus, we invariably fulfill the requirement $u'w = 0$.

If, at any stage, a w_i is negative, this implies that the i^{th} constraint is violated. In the next step, if we impose $w_i = 0$, we maximize $a'x - \frac{1}{2}x'Bx$ subject to the same constraints as before plus the i^{th} constraint. This will decrease the value of Q^*.

If, however, at any stage u_i is negative (hence $w_i = 0$), then the i^{th} constrained is imposed ($c_i'x = d_i$). In the next step, putting $u_i = 0$, we maximize $a'x - \frac{1}{2}x'Bx$ subject to the same constraints as before, minus the i^{th} constraint. This will increase the value of Q^*.

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[2] If there are n non-negativity conditions the first n elements of w are the values of x. These values are

$$x = 0 - (-I)'B^{-1}a = B^{-1}a ,$$

the vector of the unconstrained maximum.

Variables imposed to be zero are called non-basic variables. The other variables are called basic variables. In the initial solution the u_i $(i = 1, \ldots, m)$ are non-basic, the w_i $(i = 1, \ldots, m)$ are basic. Throughout all stages of the solution process, if u_i is basic, then w_i is non-basic and vice versa.

Theorem 3: Consider a solution $(\bar{u}, \bar{w})$ to (2.10) satisfying $\bar{u}'\bar{w} = 0$. Then, if $\bar{u}_i = 0$, $\bar{w}_i < 0$ a switch making u_i basic and w_i non-basic will lead to $u_i > 0$. Conversely, if $u_i < 0$, $w_i = 0$, a switch making u_i non-basic will lead to $w_i > 0$.

Proof: If $u_i < 0$, $w_i = 0$ the switch will increase the value of Q, because we are maximizing subject to one constraint less. Since

(1.11) $$\frac{dQ}{dd_i} = u_i < 0$$

increasing Q^* implies decreasing d_i, or increasing w_i (from 0 to some positive value). Again, if $w_i < 0$, $u_i = 0$, a switch will decrease the value of Q^*, because we are maximizing subject to one constraint more. Now

(1.12) $$\frac{dQ}{dw_i} = -u_i \ .$$

An increase in w_i (from a negative value to 0) decreases Q, hence increases the value of u_i (from 0 to some positive value)[3].

The question which w_i or u_k, if negative, to replace is of progmatic interest. The most natural approach appears to be to replace the most negative value. An alternative procedure is to consider all quotients w_i/u_i (all $w_i | w_i < 0$) and u_k/w_k (all $u_k | u_k < 0$) and switch the variables with the largest ratio. Neither of these procedures is foolproof against cycling, but for practical purposes this is of no consequence. For theoretical purposes, switching procedures can be (and have been) constructed which exclude cycling altogether, cf. [1].

An Example

Maximize $3x_1 + 4x_2 - 3x_1^2 - 4x_1x_2 - 1\frac{1}{2}x_2^2$

Subject to
$$
\begin{aligned}
-x_1 \quad\quad &\leq 0 \\
-x_2 &\leq 0 \\
x_1 + 2x_2 &\leq 4
\end{aligned}
$$

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[3] A little algebra shows that specifically if $u_i < 0$, $w_i = 0$, then the switch leads to a value of w_i equal to $-\frac{1}{\rho}u_i$, where ρ is the $(h, h)^{th}$ diagonal element of the inverse of the positive definite matrix $C'_w B^{-1} C_w$ if c_i is the h^{th} row of C'_w, cf. footnote 1. Conversely, if $w_i < 0$, $u_i = 0$ the value of u_i after the switch is $-\sigma w_i$, where σ is the $(k, k)^{th}$ diagonal element of the inverse of $C'_w B^{-1} C_w$, if the newly included c_i is the k^{th} row of C'_w.

Hence,

$$a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}, \quad C' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} 1\frac{1}{2} & -2 \\ -2 & 3 \end{bmatrix} \qquad C'B^{-1}C = \begin{bmatrix} 1\frac{1}{2} & -2 & 2\frac{1}{2} \\ -2 & 3 & -4 \\ 2\frac{1}{2} & -4 & 5\frac{1}{2} \end{bmatrix} \qquad d - C'B^{-1}a = \begin{bmatrix} -3\frac{1}{2} \\ 6 \\ -4\frac{1}{2} \end{bmatrix}$$

Hence, consider:

Basic	Value	u_1	u_2	u_3	w_1	w_2	w_3
w_1	$-3\frac{1}{2}$	$-1\frac{1}{2}$	2	$-2\frac{1}{2}$	1	0	0
w_2	6	2	-3	4	0	1	0
w_3	$-4\frac{1}{2}$	$-2\frac{1}{2}$	4	$-5\frac{1}{2}$	0	0	1

Switch w_3 and u_3, since w_3 is the most negative.

w_1	$-1\frac{5}{11}$	$-\frac{4}{11}$	$\frac{2}{11}$	0	1	0	$-\frac{5}{11}$
w_2	$2\frac{8}{11}$	$\frac{2}{11}$	$-\frac{1}{11}$	0	0	1	$\frac{8}{11}$
u_3	$\frac{3}{11}$	$\frac{5}{11}$	$-\frac{8}{11}$	1	0	0	$-\frac{2}{11}$

Switch w_1 and u_1

Basic	Value	u_1	u_2	u_3	w_1	w_2	w_3
u_1	4	1	$-\frac{1}{2}$	0	$-\frac{11}{4}$	0	$\frac{5}{4}$
w_2	2	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
u_3	-1	0	$-\frac{1}{2}$	1	$\frac{5}{4}$	0	$-\frac{3}{4}$

Switch u_3 and w_3:

u_1	$2\frac{1}{3}$	1	$-\frac{4}{3}$	$\frac{5}{3}$	$-\frac{2}{3}$	0	0
w_2	$1\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	1	0
w_3	$1\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	0	1

This is the solution. Knowing the vector w we can easily solve $C'x + w = d$; in fact, when there are non-negativity conditions the first n elements of w coincide with those of x; hence $x_1 = 0$, $x_2 = 1\frac{1}{3}$. We would have gotten the final tableau directly by switching w_1 and u_1 in the first tableau $(\frac{w_1}{u_1} > \frac{w_3}{u_3})$. We also have $u_1 = 2\frac{1}{3}$, $u_2 = 0$, $u_3 = 0$.

These values of the Lagrangeans are an indication of the value of the sources, by virtue of (1.6). Moreover, they are the solution to the so-called dual problem. Formally, if the primal problem is given by (1.1) - (1.2), and has a solution x^*, say, then the m-vector u^* solves the problem

$$\text{(1.13)} \qquad \text{Minimize} \quad d'u + \frac{1}{2}x^{*\prime}Bx^*$$

subject to

$$\text{(1.14)} \qquad u \geq 0$$

and

$$\text{(1.15)} \qquad Cu + Bx^* = a \ .$$

Moreover:

$$\text{(1.16)} \qquad a'x^* - \frac{1}{2}x^{*\prime}Bx^* = d'u^* + \frac{1}{2}x^{*\prime}Bx^* \ .$$

Relations (1.14) - (1.16) can be verified numerically.

The example is illustrated in Figure 1 .

I : $x_1 = -3\frac{1}{2}$; $x_2 = 6$; $\varphi = 6\frac{3}{4}$

II : $x_1 = -1\frac{5}{11}$; $x_2 = 2\frac{8}{11}$; $\varphi = 4\frac{10}{11}$

III : $x_1 = 0$; $x_2 = 2$; $\varphi = 2$

IV : $x_1 = 0$; $x_2 = 1\frac{1}{3}$; $\varphi = 2\frac{2}{3}$

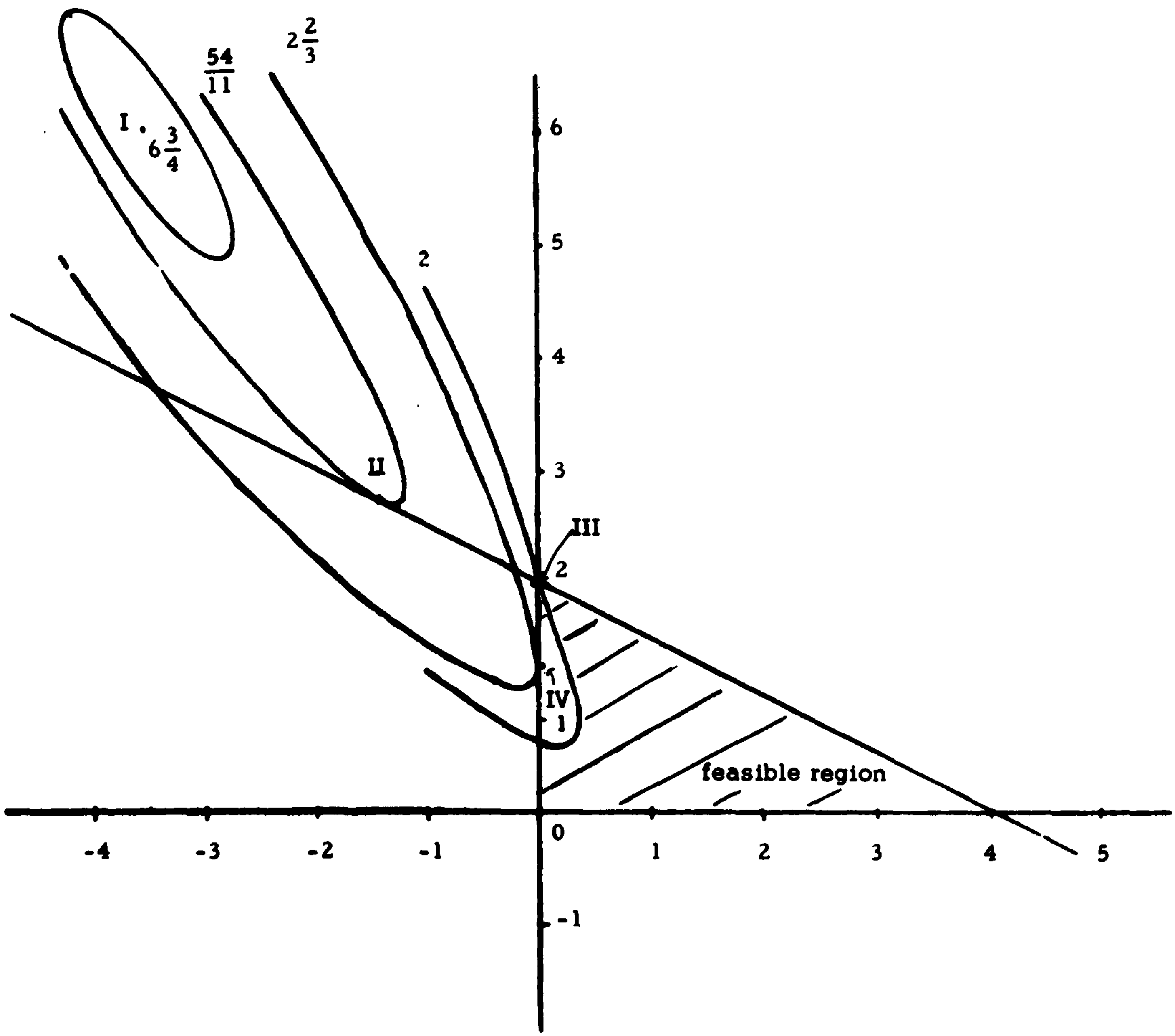

Figure 1

2. Linear Programming

The easy procedure outlined above breaks down when B is semi-definite. For then $C'B^{-1}C$ and $C'B^{-1}a$ cannot be determined. Consider first the most extreme case, where the B-matrix is the null-matrix, and hence we have a linear programming problem. We can then apply the procedure above by introducing a large value λ and considering the problem:

$$(2.1) \qquad \text{Maximize} \qquad \lambda(a'x) - \frac{1}{2}x'Ix$$

subject to

$$(2.2) \qquad C'x \leq d \ ,$$

which is clearly equivalent to the linear programming problem of maximizing $a'x$ subject to $C'x \leq d$.

As an illustration consider the problem:

$$\text{Maximize} \qquad L(x) = 3x_1 + ux_2 \qquad [\text{or } \varphi(\lambda x) = \lambda(3x_1 + 4x_2) - \frac{1}{2}x'Ix]$$

$$\begin{aligned} \text{Subject to} \qquad -x_1 \qquad &\leqq 0 \\ -x_2 &\leqq 0 \\ x_1 - 2x_2 &\leqq 1 \\ 2x_1 - 3x_2 &\leqq 2 \\ x_1 + 4x_2 &\leqq 3 \end{aligned}$$

We obtain, since $B = B^{-1} = I$:

$$C'B^{-1}C = C'C = \begin{bmatrix} 1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 2 & 3 & -4 \\ -1 & 2 & 5 & 8 & -7 \\ -2 & 3 & 8 & 13 & -10 \\ -1 & -4 & -7 & -10 & -17 \end{bmatrix} \quad \text{and} \quad d - C'B^{-1}a = d - C'a = \begin{bmatrix} 0 + 3\lambda \\ 0 + 4\lambda \\ 1 + 5\lambda \\ 2 + 5\lambda \\ 3 - 19\lambda \end{bmatrix}$$

which leads to the following tableaux:

I

Basic	Value	u_1	u_2	u_3	u_4	u_5	w_1	w_2	w_3	w_4	w_5
w_1	$0 + 3\lambda$	-1	0	1	2	1	1	0	0	0	0
w_2	$0 + 4\lambda$	0	-1	-2	-3	4	0	1	0	0	0
w_3	$1 + 5\lambda$	1	-2	-5	-8	7	0	0	1	0	0
w_4	$2 + 6\lambda$	2	-3	-8	-13	10	0	0	0	1	0
w_5	$3 - 19\lambda$	1	4	7	10	-17	0	0	0	0	1

II (Switch w_5 and u_5; divide all entries by 17)

w_1	$3 + 32\lambda$	-16	4	24	44	0	17	0	0	0	1
w_2	$12 - 8\lambda$	4	-1	-6	11	0	0	17	0	0	4
w_3	$38 - 48\lambda$	24	-6	-36	-66	0	0	0	17	0	7
w_4	$64 - 88\lambda$	44	-11	-66	-121	0	0	0	0	17	10
u_5	$-3 + 19\lambda$	-1	-4	-7	-10	17	0	0	0	0	-1

III (Switch w_4 and u_4)

Basic	Value	u_1	u_2	u_3	u_4	u_5	w_1	w_2	w_3	w_4	w_5
w_1	$\frac{17}{11} + 0\lambda$				0						
w_2	$\frac{4}{11} + 0\lambda$				0						
w_3	$\frac{2}{11} + 0\lambda$	irrelevant			0			irrelevant			
u_4	$-\frac{64}{14} + \frac{8}{11}\lambda$				1						
u_5	$-\frac{59}{121} + \frac{17}{11}\lambda$				0						

The solution therefore is: $w_1 = x_1 = \frac{17}{11}$, $w_2 = x_2 = \frac{4}{11}$, $w_3 = \frac{2}{11}$, $w_4 = w_5 = 0$, $\varphi = \frac{67}{11}$. The solution is illustrated in Figure 2. The first tableau gives as solution $(3\lambda, 4\lambda)$, i.e. a point on the gradient line to $3x_1 + 4x_2$, the linear part of the objective function. Since this point violates the 5^{th} constraint, we next maximize $3\lambda x_1 + 4\lambda x_2 - \frac{1}{2}x'Ix$ subject to $x_1 + 4x_2 = 3$. Because maximizing $3\lambda x_1 + 4\lambda x_2 - \frac{1}{2}x'Ix$ is the same as minimizing $\frac{1}{2}(x_1 - 3\lambda,\ x_2 - 4\lambda)\, I \begin{bmatrix} x_1 - 3\lambda \\ x_2 - 4\lambda \end{bmatrix}$, we actually minimize the distance between $(3\lambda, 4\lambda)$ and the line $x_1 + 4x_2 = 3$; i.e. we find the projection of $(3\lambda, 4\lambda)$ on $x_1 + 4x_2 = 3$. In the next tableau we find the solution. Either criterion tells us to switch u_4 with w_4.

It is of some interest to notice that it will take at least n steps to get to the solution point, since at least n elements of w will be zero in the solution - barring infinite maxima. It may also be observed that the solution of the dual is given by the coefficients of λ in the final tableau, i.e. $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = \frac{8}{11}$, $u_5 = \frac{17}{11}$. This follows, because $d\varphi(\lambda x)/dx = \lambda u$.

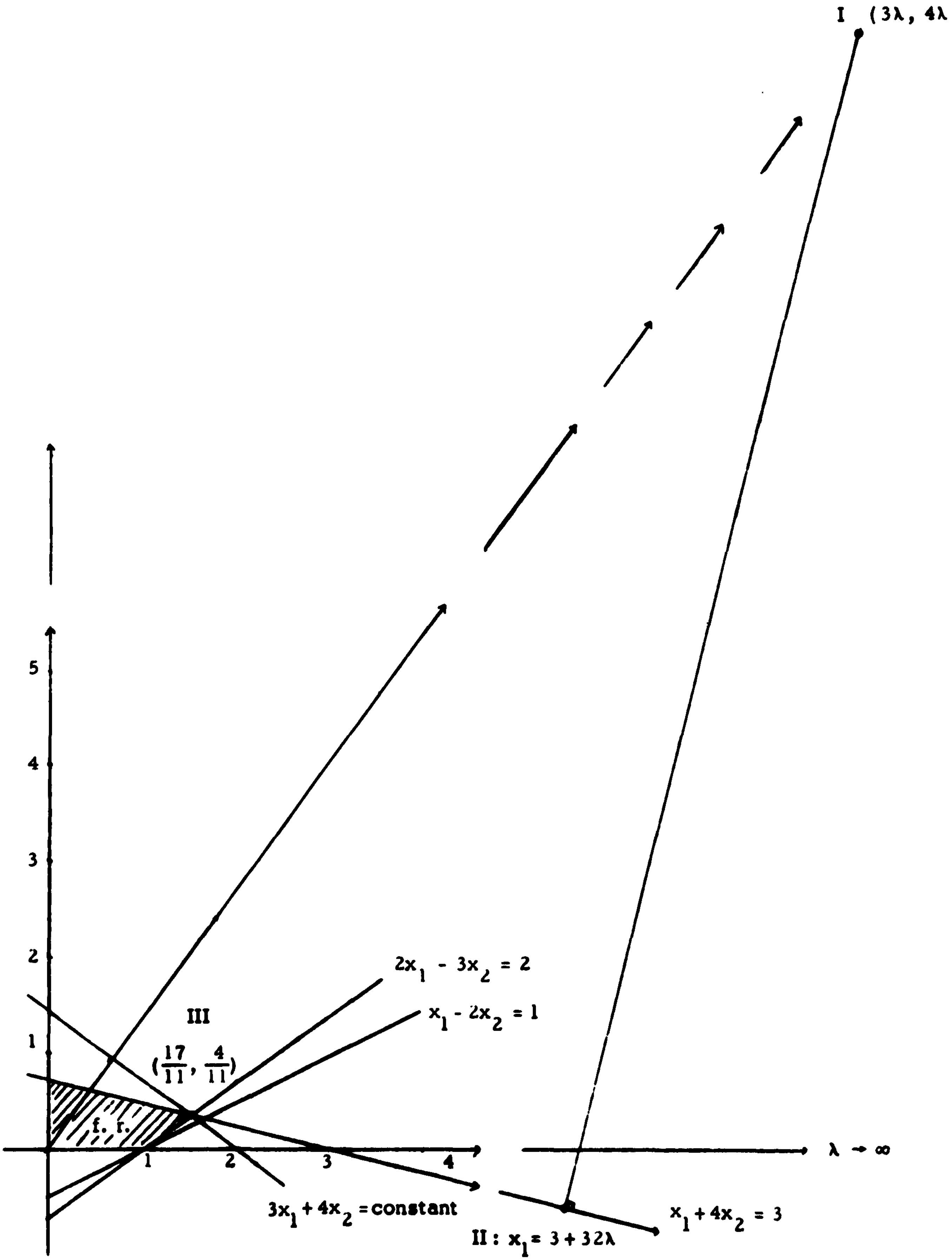
I (3λ, 4λ)
5
4
3
2
1
III
$(\frac{17}{11}, \frac{4}{11})$
f. r.
1
2
3
4
λ → ∞
$2x_1 - 3x_2 = 2$
$x_1 - 2x_2 = 1$
$3x_1 + 4x_2 =$ constant
$x_1 + 4x_2 = 3$
II : $x_1 = 3 + 32\lambda$
$x_2 = 12 - 8\lambda$

Figure 2

3. Singular Quadratic Programming

We are now in a position to consider the more general case of quadratic programming with a singular quadratic part. Quite generally, if B is a positive semi-definite matrix of order $n \times n$ and rank r there exists an $n \times r$ matrix T' such that $T'T = B$. Introducing the transformation

$$(3.1) \qquad y = \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} \begin{matrix} = \\ \\ = \end{matrix} \begin{bmatrix} Tx \\ \\ Ex \end{bmatrix} = T^* x$$

[where y_1 is a r-vector, y_2 a $(n-r)$-vector and E a matrix which has as k^{th} row the $(r+k)^{th}$ n-dimensional unit vector] we can therefore write, provided we take care that the first r columns of T are independent;

$$(3.2) \qquad a'x - \frac{1}{2} x'Bx = a'T^{*-1}y - \frac{1}{2} y_1' I y_1 \quad .$$

Maximizing $a'T^{*-1}y - \frac{1}{2} y_1' I y_1$ is equivalent to maximizing

$$\lambda (a'T^{*-1}y - \frac{1}{2} y_1' I y_1) - \frac{1}{2} y_2' I y_2 \ ,$$

provided λ is very large.

As an example consider the problem

Maximize $3x_1 + 4x_2 + x_3 - \frac{1}{2}(x_1\ x_2\ x_3)\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

Subject to

$$\begin{aligned} -x_1 \quad\quad\quad\quad &\leqq 0 \\ -x_2 \quad\quad &\leqq 0 \\ -x_3 &\leqq 0 \\ x_1 + 2x_2 + x_3 &\leqq 4 \end{aligned}$$

Using $y = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$ or $x = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} y$

We reformulate this problem as follows.

Maximize $\lambda(3y_1 - 2y_2 + 4y_3 - \frac{1}{2}y_1^2) - \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2$

Subject to

$$\begin{aligned} -y_1 + 2y_2 - y_3 &\leqq 0 \\ -y_2 \quad\quad &\leqq 0 \\ -y_3 &\leqq 0 \\ y_1 \quad\quad + 2y_3 &\leqq 4 \end{aligned}$$

With
$$C' = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \qquad B^* = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \qquad a^* = \begin{bmatrix} 3\lambda \\ -2\lambda \\ 4\lambda \end{bmatrix}$$

we obtain

$$-C'B^{-1}C = \begin{bmatrix} -\frac{5\lambda+1}{\lambda} & 2 & -1 & \frac{2\lambda+1}{\lambda} \\ 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 2 \\ \frac{2\lambda+1}{\lambda} & 0 & 2 & -\frac{4\lambda+1}{\lambda} \end{bmatrix} \qquad d - C'B^{*-1}a^* = \begin{bmatrix} 3+8\lambda \\ -2\lambda \\ 4\lambda \\ 1-8\lambda \end{bmatrix}$$

Thus, we get the following tableaux:

I

Basis	Value	u_1	u_2	u_3	u_4	w_1	w_2	w_3	w_4
w_1	$3+8\lambda$	$-\frac{5\lambda+1}{\lambda}$	2	-1	$\frac{2\lambda+1}{\lambda}$	1	0	0	0
w_2	-2λ	2	-1	0	0	0	0	0	0
w_3	4λ	-1	0	-1	2	0	0	0	1
w_4	$1-8\lambda$	$\frac{2\lambda+1}{\lambda}$	0	2	$-\frac{4\lambda+1}{\lambda}$	0	0	0	1

II (With, for reasons of simplicity, u_2 replacing w_2)

Basis	Value	u_1	u_2	u_3	u_4	w_1	w_2	w_3	w_4
w_1	$3+4\lambda$	$-\frac{\lambda+1}{\lambda}$	0	-1	$\frac{2\lambda+1}{\lambda}$	1	2	0	0
u_2	2λ	-2	1	0	0	0	-1	0	0
w_3	4λ	-1	0	-1	2	0	0	1	0
w_4	$1-8\lambda$	$\frac{2\lambda+1}{\lambda}$	0	2	$-\frac{4\lambda+1}{\lambda}$	0	0	0	1

III (Switching u_4 and w_4)

w_1	$\frac{10\lambda+4}{4\lambda+1}$	$-\frac{1}{4\lambda+1}$	0	$\frac{1}{4\lambda+1}$	0	1	2	0	$\frac{2\lambda+1}{4\lambda+1}$
u_2	2λ	-2	1	0	0	0	-1	1	0
w_3	$\frac{6\lambda}{4\lambda+1}$	$\frac{1}{4\lambda+1}$	0	$-\frac{1}{4\lambda+1}$	0	0	0	0	$\frac{2\lambda}{4\lambda+1}$
u_4	$\frac{\lambda(8\lambda-1)}{4\lambda+1}$	$-\frac{2\lambda+1}{4\lambda+1}$	0	$-\frac{2\lambda}{4\lambda+1}$	1	0	0	0	$-\frac{\lambda}{4\lambda+1}$

In the first tableau we have $y_2 = w_2 = -2\lambda$; $y_3 = w_3 = 4\lambda$; $y_1 = 2y_2 - y_3 + w_1 = 3$. This is clearly the solution maximizing $3y_1 - 2y_2 + 4y_3 - \frac{1}{2}y_1^2$. However, we violate the 2^{nd} and 4^{th} constraint (as indicated by the negative values for w_2 and w_4). First imposing $w_2 = 0$ and next $w_4 = 0$ leads to the final tableau. We have $w_1 = \frac{10\lambda + y}{4\lambda + 1}$, $w_2 = 0$, $w_3 = \frac{6\lambda}{4\lambda + 1}$ and $w_4 = 0$.

Hence $y_2 = 0$, $y_3 = \frac{6\lambda}{4\lambda + 1} = 1\frac{1}{2}$, $y_1 = \frac{4\lambda + 4}{4\lambda + 1} = 1$, and, as a check, $y_1 + 2y_3 = 4$. Transforming back to the original variables we find $x_1 = 2\frac{1}{2}$, $x_2 = 0$, $x_3 = 1\frac{1}{2}$, and the value of the objective function equals $8\frac{1}{2}$. Again, the coefficients of λ give the dual values; hence $u_1 = 0$, $u_2 = 2$, $u_3 = 0$, $u_4 = \frac{8\lambda - 1}{4\lambda + 1} = 2$. Checking (1.15) we find:

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

Writing $\ell(\lambda)$ for linear functions in λ, and $q(\lambda)$ for quadratic functions of λ, any problem with a finite solution will have forms of the structure $\frac{\ell_1(\lambda)}{\ell_2(\lambda)}$ for all basic w_i, and $\frac{q(\lambda)}{\ell(\lambda)}$ for basic u_i. The expressions in the tableaux themselves will invariably be of the form $\frac{\ell_3(\lambda)}{\ell_4(\lambda)}$.

Again, we can make the observation that, if there is a finite solution, at least $n - r$ steps will be required. For:

Theorem 4: If the maximum of $a'x - \frac{1}{2}x'Bx$, (where B is of order $n \times n$ and rank r), subject to $C'x \leq d$ is finite, then at least $n - r$ constraints will be exactly satisfied.

Proof: Rewrite so as to get

Maximize $\quad a'T^{*-1}y - \frac{1}{2} y_1' I y_1$

Subject to $\quad C'T^{*-1}y \leqq d \quad .$

The r variables y_1 take finite values. For any set of values, say $\bar{y}_1$, there remains a linear programming problem in $n-r$ variables. Hence at least $n-r$ constraints will be binding.

If we can find $n-r$ constraints binding in the solution point by inspection, the problem can immediately be transformed to a non-singular quadratic programming problem. For maximizing $a'x - \frac{1}{2} x'Bx$ (B is of order $n \times n$, rank r) subject to $C'x = d$ (where C' has at least $n-r$ rows c_h') is equivalent to maximizing

$$a'x - \frac{1}{2} x'[B + c_1 c_1' + \dots c_{n-r} c_{n-r}'] x$$

subject to $C'x = d$. The matrix $B + c_1 c_1' + \dots + c_{n-r} c_{n-r}'$ will be of full rank iff the matrix $\begin{bmatrix} T \\ C' \end{bmatrix}$ is of full rank.

REFERENCES

[1] C. Van de Panne, "A Dual Method for Quadratic Programming", (unpublished report of the Carnegie Institute of Technology, Graduate School of Industrial Administration).